COVARIANT COMPLETELY POSITIVE LINEAR MAPS BETWEEN LOCALLY C^*-ALGEBRAS

MARIA JOIŢA*†

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Abstract

We prove a covariant version of the KSGNS (Kasparov, Stinespring, Gel'fand,Naimark,Segal) construction for completely positive linear maps between locally C^*-algebras. As an application of this construction, we show that a covariant completely positive linear map ρ from a locally C^*-algebra A to another locally C^* -algebra B with respect to a locally C^*-dynamical system (G, A, α) extends to a completely positive linear map on the crossed product $A \times_\alpha G$.

1 Introduction

Locally C^*-algebras are generalizations of C^*-algebras. Instead of being given by a single norm, the topology on a locally C^*-algebra is defined by a directed family of C^*-seminorms. Such important concepts as Hilbert C^*-modules, adjointable operators, (completely) positive linear maps, C^*-dynamical systems can be defined with obvious modifications in the framework of locally C^*-algebras. The proofs are not always straightforward.

It is well-known that a positive linear functional on a C^*-algebra A induces a representation of this C^*-algebra on a Hilbert space by the GNS (Gel'fand, Naimark, Segal) construction (see, for example, [**1**]) . Stinespring [**13**] extends this construction for completely positive linear map from A to $L(H)$, the C^*-algebra of all bounded linear operators on a Hilbert space H. On the other

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hand, Paschke [**8**] (respectively, Kasparov [**5**]) shows that a completely positive linear map from A to another C^*-algebra B (respectively, from A to the C^*-algebra of all adjointable operators on the Hilbert C^*-module H_B) induces a representation of A on a Hilbert B -module. In [**2**], the author extends the KSGNS (Kasparov, Stinespring, Gel'fand, Naimark, Segal) construction for a strict continuous, completely positive linear map from a locally C^*-algebra A to $L_B(E)$, the locally C^*-algebra of all adjointable operators on a Hilbert module E over a locally C^*-algebra B. In this paper we propose to prove a covariant version of this construction. Thus we show that a covariant completely positive linear map from A to $L_B(E)$ with respect to a locally C^*-dynamical system (G, A, α) induces a non-degenerate, covariant representation of (G, A, α) on a Hilbert B -module which is unique up to unitary equivalence, Theorem 3.6. Using the analog of the covariant version of Stinespring construction [**9**] for bounded operators on Hilbert C^*-modules, Kaplan [**4**] shows that a discrete covariant completely positive map ρ from a unital C^*-algebra A to another unital C^*-algebra B extends to a completely positive map from the crossed product $A \times_\alpha G$ to B. We extend this result showing that a non-degenerate, covariant, continuous completely positive linear map from a locally C^*-algebra to another locally C^*-algebra B extends to a non-degenerate, continuous completely positive linear map on the crossed product $A \times_\alpha G$, Proposition 3.9.

2 Preliminaries

A locally C^*-algebra is a complete complex Hausdorff topological $*$ -algebra whose topology is determined by a directed family of C^*-seminorms. If A is a locally C^*-algebra and $S(A)$ is the set of all continuous C^*-seminorms on A, then for each $p \in S(A)$, $A_p = A/\ker p$ is a C^*-algebra in the norm induced by p, and $\{A_p; \pi_{pq}\}_{p,q \in S(A), p \geq q}$, where π_{pq} is the canonical morphism from A_p onto A_q defined by $\pi_{pq}(a + \ker p) = a + \ker q$ for all $a \in A$ is an inverse system of C^*-algebras. Moreover, A can be identified with $\lim\limits_{\leftarrow p} A_p$. The canonical map from A onto A_p is denoted by π_p.

An approximate unit of A is an increasing net $\{e_\lambda\}_{\lambda \in \Lambda}$ of positive elements of A such that $p(e_\lambda) \leq 1$ for all $p \in S(A)$ and for all $\lambda \in \Lambda$, $p(ae_\lambda - a) \to 0$ and $p(e_\lambda a - a) \to 0$ for all $p \in S(A)$ and for all $a \in A$. Any locally C^*-algebra has an approximate unit [**11**, Proposition 3.11].

A morphism of locally C^*-algebras is a continuous $*$ -morphism from a locally C^*-algebra A to another locally C^*-algebra B. An isomorphism of locally C^*-algebras from A to B is a bijective map $\Phi : A \to B$ such that Φ and Φ^{-1} are

morphisms of locally C^*-algebras.

Let $M_n(A)$ denote the $*$ -algebra of all $n \times n$ matrices over A with the algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of A. Then $\{M_n(A_p); \pi_{pq}^{(n)}\}_{p,q\in S(A),p\geq q}$, where $\pi_{pq}^{(n)}([\pi_p(a_{ij})]_{i,j=1}^n) = [\pi_q(a_{ij})]_{i,j=1}^n$, is an inverse system of C^*-algebras and $M_n(A)$ can be identified with $\lim\limits_{\leftarrow p} M_n(A_p)$.

A linear map $\rho : A \to B$ between two locally C^*-algebras is completely positive if the linear maps $\rho^{(n)} : M_n(A) \to M_n(B)$ defined by

$$\rho^{(n)}([a_{ij}]_{i,j=1}^n) = [\rho(a_{ij})]_{i,j=1}^n$$

$n = 1, 2, ..., n, ...,$ are all positive.

Definition 2.1 A *pre-Hilbert A-module* is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ which is $\mathbb{C}$ -and A-linear in its second variable and satisfies the following relations:

1. (a) i. $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$;
   ii. $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$;
   iii. $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

We say that E is a Hilbert A-module if E is complete with respect to the topology determined by the family of seminorms $\{\|\cdot\|_p\}_{p\in S(A)}$ where $\|\xi\|_p = \sqrt{p(\langle \xi, \xi \rangle)}, \xi \in E$ [11, Definition 4.1].

Let E be a Hilbert A-module. For $p \in S(A)$, $\mathcal{E}_p = \{\xi \in E; p(\langle \xi, \xi \rangle) = 0\}$ is a closed submodule of E and $E_p = E/\mathcal{E}_p$ is a Hilbert A_p-module with $(\xi + \mathcal{E}_p)\pi_p(a) = \xi a + \mathcal{E}_p$ and $\langle \xi + \mathcal{E}_p, \eta + \mathcal{E}_p \rangle = \pi_p(\langle \xi, \eta \rangle)$. The canonical map from E onto E_p is denoted by σ_p. For $p, q \in S(A)$, $p \geq q$ there is a canonical morphism of vector spaces σ_{pq} from E_p onto E_q such that $\sigma_{pq}(\sigma_p(\xi)) = \sigma_q(\xi)$, $\xi \in E$. Then $\{E_p; A_p; \sigma_{pq}\}_{p,q\in S(A),p\geq q}$ is an inverse system of Hilbert C^*-modules in the following sense: $\sigma_{pq}(\xi_p a_p) = \sigma_{pq}(\xi_p)\pi_{pq}(a_p), \xi_p \in E_p, a_p \in A_p$; $\left\langle \sigma_{pq}(\xi_p), \sigma_{pq}(\eta_p) \right\rangle = \pi_{pq}(\left\langle \xi_p, \eta_p \right\rangle), \xi_p, \eta_p \in E_p$; $\sigma_{pp}(\xi_p) = \xi_p$, $\xi_p \in E_p$ and $\sigma_{qr} \circ \sigma_{pq} = \sigma_{pr}$ if $p \geq q \geq r$, and $\lim\limits_{\leftarrow p} E_p$ is a Hilbert A-module which can be identified with E [**11**, Proposition 4.4].

Let E and F be Hilbert A-modules. We say that an A-module morphism $T : E \to F$ is adjointable if there is an A-module morphism $T^* : F \to E$ such that $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$ for every $\xi \in E$ and $\eta \in F$. Any adjointable A-module morphism is continuous. The set $L_A(E, F)$ of all adjointable A-module

morphisms from E into F becomes a locally convex space with topology defined by the family of seminorms $\{\widetilde{p}\}_{p\in S(A)}$, where $\widetilde{p}(T)=\|(\pi_p)_*(T)\|_{L_{A_p}(E_p,F_p)}$, $T\in L_A(E,F)$ and $(\pi_p)_*(T)(\xi+\mathcal{E}_p)=T\xi+\mathcal{F}_p$, $\xi\in E$. Moreover, $\{L_{A_p}(E_p,F_p);\ (\pi_{pq})_*\}_{p,q\in S(A),p\geq q}$, where $(\pi_{pq})_*:L_{A_p}(E_p,F_p)\to L_{A_q}(E_q,F_q)$, $(\pi_{pq})_*(T_p)(\sigma_q(\xi))=\chi_{pq}(T_p(\sigma_p(\xi)))$, and χ_{pq}, $p,q\in S(A)$, $p\geq q$ are the connecting maps of the inverse system $\{F_p\}_{p\in S(A)}$, is an inverse system of Banach spaces, and $\lim\limits_{\leftarrow p} L_{A_p}(E_p,F_p)$ can be identified with $L_A(E,F)$ [**11**, Proposition 4.7]. Thus topologized, $L_A(E,E)$ becomes a locally C^*-algebra, and we write $L_A(E)$ for $L_A(E,E)$.

The strict topology on $L_A(E)$ is defined by the family of seminorms $\{\|\cdot\|_{p,\xi}\}_{(p,\xi)\in S(A)\times E}$, where $\|T\|_{p,\xi}=\|T\xi\|_p+\|T^*\xi\|_p$, $T\in L_A(E)$.

Two Hilbert A -modules E and F are unitarily equivalent if there is a unitary element in $L_A(E,F)$.

A non-degenerate representation of a locally C^*-algebra A on a Hilbert module E over a locally C^*-algebra B is a morphism of locally C^*-algebras Φ from A to $L_B(E)$ such that $\Phi(A)E$ is dense in E.

A continuous completely positive linear map ρ from A to $L_B(E)$ is non-degenerate if the net $\{\rho(e_\lambda)\}_{\lambda\in\Lambda}$ converges strictly to the identity map on E for some approximate unit $\{e_\lambda\}_{\lambda\in\Lambda}$ for A.

Let G be a locally compact group and let A be a locally C^*-algebra. An action of G on A is a morphism α from G to $\mathrm{Aut}(A)$, the set of all isomorphisms of locally C^*-algebras from A to A. The action α is continuous if the function $(t,a)\to\alpha_t(a)$ from $G\times A$ to A is jointly continuous. An action α is called an inverse limit action if we can write A as inverse limit $\lim\limits_{\leftarrow\delta}A_\delta$ of C^*-algebras in such a way that there are actions $\alpha^{(\delta)}$ of G on A_δ such that $\alpha_t=\lim\limits_{\leftarrow\delta}\alpha_t^{(\delta)}$ for all t in G [**12**, Definition 5.1]. An action α of G on A is a continuous inverse limit action if there is a cofinal subset $S_G(A,\alpha)$ of G-invariant continuous C^*-seminorms on A (a continuous C^*-seminorm p on A is G -invariant if $p(\alpha_t(a))=p(a)$ for all a in A and for all t in G). So if α is a continuous inverse limit action of G on A we can suppose that $S(A)=S_G(A,\alpha)$.

A locally C^*-dynamical system is a triple (G,A,α), where G is a locally compact group, A is a locally C^*-algebra and α is a continuous action of G on A.

Let α be a continuous inverse limit action of G on A. The set $C_c(G,A)$ of all continuous functions from G to A with compact support becomes a $*$ -algebra

with convolution of two functions

$$(f \times h)(s) = \int_G f(t)\alpha_t\left(h(t^{-1}s)\right) dt$$

as product and involution defined by

$$f^\sharp(t) = \Delta(t)^{-1}\alpha_t\left(f(t^{-1})^*\right)$$

where Δ is the modular function on G. The Hausdorff completion of $C_c(G, A)$ with respect to the topology defined by the family of submultiplicative $*$ -seminorms $\{N_p\}_{p \in S(A)}$, where

$$N_p(f) = \int_G p(f(s))ds$$

is denoted by $L^1(G, A, \alpha)$ and the enveloping locally C^*-algebra $A \times_\alpha G$ of $L^1(G, A, \alpha)$ is called the crossed product of A by α [**3**, Definition 3.14]. Moreover, the C^*-algebras $(A \times_\alpha G)_p$ and $A_p \times_{\alpha^{(p)}} G$ can be identified for each $p \in S(A)$ and so $A \times_\alpha G$ can be identified with $\lim\limits_{\leftarrow p} A_p \times_{\alpha^{(p)}} G$ [**3**, Remark 3.15].

3 Covariant representations associated with a covariant completely positive linear map

Let B be a locally C^*-algebra, let E be a Hilbert B -module and let G be a locally compact group.

Definition 3.1 A *unitary representation* of G on E is a map u from G to $L_B(E)$ such that

1. *(a)* u_g is a unitary element in $L_B(E)$ for all $g \in G$;
   (b) $u_{gt} = u_g u_t$ for all $g, t \in G$;
   (c) the map $g \mapsto u_g\xi$ from G to E is continuous for all $\xi \in E$.

Remark 3.2 *If u is a unitary representation of G on E, then for each $q \in S(B)$, $g \mapsto (\pi_q)_* \circ u$ is a unitary representation of G on E_q. Moreover, $u_g = \lim\limits_{\leftarrow q} u_g^{(q)}$, where $u_g^{(q)} = (\pi_q)_*(u_g)$, for all $g \in G$.*

Definition 3.3 *A non-degenerate, covariant representation* of a locally C^*-dynamical system (G, A, α) on a Hilbert B -module E is a triple (Φ, v, E), where Φ is a non-degenerate representation of A on E, v is a unitary representation of G on E and

$$\Phi(\alpha_g(a)) = v_g \Phi(a) v_g^*$$

for all $g \in G$ and $a \in A$.

Proposition 3.4 *Let (G, A, α) be a locally C^*-dynamical system such that α is an inverse limit action, let (Φ, v, E) be a non-degenerate covariant representation of (G, A, α) on a Hilbert B -module E. Then there is a unique non-degenerate representation $\Phi \times v$ of the crossed product $A \times_\alpha G$ on E such that*

$$(\Phi \times v)(f) = \int_G \Phi(f(g)) v_g dg$$

for all $f \in C_c(G, A)$.

PROOF. We partition the proof into two steps.

Step1. We suppose that B is a C^*-algebra.

Since Φ is a representation of A on E, there is $p \in S(A)$ such that $\|\Phi\,(a)\|_{L_B(E)} \leq p(a)$ for all $a \in A$. From this fact, we deduce that there is a morphism of C^*-algebras Φ_p from A_p to $L_B(E)$ such that $\Phi_p \circ \pi_p = \Phi$. Therefore Φ_p is a representation of A_p on E, and moreover, it is non-degenerate, since Φ is non-degenerate and π_p is surjective. It is not difficult to check that (Φ_p, v, E) is a non-degenerate covariant representation of $(G, A_p, \alpha^{(p)})$. Then there is a unique non-degenerate representation $\Phi_p \times v$ of $A_p \times_{\alpha^{(p)}} G$ on E such that

$$(\Phi_p \times v)(f) = \int_G \Phi_p(f(g)) v_g dg$$

for all $f \in C_c(G, A_p)$ (see, for example, Proposition 7.6.4, [**10**]). Therefore $\Phi \times v = (\Phi_p \times v) \circ \widetilde{\pi}_p$, where $\widetilde{\pi}_p$ is the canonical map from $A \times_\alpha G$ onto $A_p \times_{\alpha^{(p)}} G$ is a non-degenerate representation of $A \times_\alpha G$ on E such that

$$(\Phi \times v)(f) = \int_G \Phi_p(\widetilde{\pi}_p(f)(g)) v_g dg = \int_G \Phi(f(g)) v_g dg$$

for all $f \in C_c(G, A)$, and since $C_c(G, A)$ is dense in $A \times_\alpha G$, $\Phi \times v$ is unique with the above property.

Step2. The general case.

For each $q \in S(B)$, $(\pi_q)_* \circ \Phi$ is a non-degenerate representation of A on E_q, and $((\pi_q)_* \circ \Phi, v^{(q)}, E_q)$ is a non-degenerate covariant representation of (G, A, α) on E_q. By Step 1 there is a unique non-degenerate representation $((\pi_q)_* \circ \Phi) \times v^{(q)}$ of $A \times_\alpha G$ on E_q such that

$$(((\pi_q)_* \circ \Phi) \times v^{(q)})(f) = \int\limits_G (\pi_q)_*(\Phi(f(g)))v_g^{(q)} dg$$

for all $f \in C_c(G, A)$. By Lemma 3.7 in [**3**], we have

$$\begin{aligned}(\pi_{qr})_*((((\pi_q)_* \circ \Phi) \times v^{(q)})(f)) &= \int\limits_G (\pi_{qr})_*((\pi_q)_*(\Phi(f(g)))v_g^{(q)})dg \\ &= \int\limits_G (\pi_r)_*(\Phi(f(g)))v_g^{(r)})dg \\ &= (((\pi_r)_* \circ \Phi) \times v^{(r)})(f)\end{aligned}$$

for all $f \in C_c(G, A)$ and for all $q, r \in S(B)$ with $q \geq r$. Therefore $(\pi_{qr})_* \circ (((\pi_q)_* \circ \Phi) \times v^{(q)}) = ((\pi_r)_* \circ \Phi) \times v^{(r)}$ for all $q, r \in S(B)$ with $q \geq r$. This implies that there is a continuous $*$ -morphism $\Phi \times v$ from $A \times_\alpha G$ to $L_B(E)$ such that $(\pi_q)_* \circ (\Phi \times v) = ((\pi_q)_* \circ \Phi) \times v^{(q)}$ for all $q \in S(B)$. Using Lemma III 3.1 in [**7**], it is not hard to check that $(\Phi \times v)(A \times_\alpha G)E$ is dense in E. Therefore $\Phi \times v$ is a non-degenerate representation of $A \times_\alpha G$ on E, and by Lemma 3.7 in [**3**],

$$(\Phi \times v)(f) = \int\limits_G \Phi(f(g))v_g dg$$

for all $f \in C_c(G, A)$. Moreover, since $C_c(G, A)$ is dense in $A \times_\alpha G$, $\Phi \times v$ is unique with the above property. q.e.d.

Definition 3.5 Let (G, A, α) be a locally C^*-dynamical system and let u be a unitary representation of G on a Hilbert B -module E. We say that a completely positive linear map ρ from A to $L_B(E)$ is u *-covariant* with respect to the locally C^*-dynamical system (G, A, α) if

$$\rho(\alpha_g(a)) = u_g \rho(a) u_g^*$$

for all $a \in A$ and for all $g \in G$.

Recall that if ρ is a completely positive linear map from a C^*-algebra A to $L_B(E)$, the C^*-algebra of all adjointable operators on a Hilbert module E

over a C^*-algebra B, the quotient vector space $(A \otimes_{\text{alg}} E)/\mathcal{N}_\rho$, where $\mathcal{N}_\rho = \{\sum_{i=1}^n a_i \otimes \xi_i;\ \sum_{i,j=1}^n \langle \xi_i, \rho(a_i^* a_j)\xi_j \rangle = 0\}$ becomes a pre-Hilbert B -module with the action of B on $A \otimes_{\text{alg}} E$ defined by $\left(a \otimes \xi + \mathcal{N}_\rho\right) b = a \otimes \xi b + \mathcal{N}_\rho$ and the inner-product defined by

$$\left\langle \sum_{i=1}^n a_i \otimes \xi_i + \mathcal{N}_\rho, \sum_{j=1}^m b_j \otimes \eta_j + \mathcal{N}_\rho \right\rangle_\rho = \sum_{i=1}^n \sum_{j=1}^m \langle \xi_i, \rho(a_i^* b_j)\eta_j \rangle .$$

The following theorem is a covariant version of Theorem 4.6 in [**2**].

Theorem 3.6 *Let (G, A, α) be a locally C^*-dynamical system, let u be a unitary representation of G on a Hilbert module E over a locally C^*-algebra B, and let ρ be a u -covariant, non-degenerate, continuous completely positive linear map from A to $L_B(E)$.*

1. *Then there is a covariant representation $(\Phi_\rho, v^\rho, E_\rho)$ of (G, A, α) and an element V_ρ in $L_B(E, E_\rho)$ such that*
   (a) *$\rho(a) = V_\rho^* \Phi_\rho(a) V_\rho$ for all $a \in A$;*
   (b) *$\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in E\}$ spans a dense submodule of E_ρ;*
   (c) *$v_g^\rho V_\rho = V_\rho u_g$ for all $g \in G$.*
2. *If F is a Hilbert B -module, (Φ, v, F) is a covariant representation of (G, A, α) and W is an element in $L_B(E, F)$ such that*
   (a) *$\rho(a) = W^* \Phi(a) W$ for all $a \in A$;*
   (b) *$\{\Phi(a)W\xi; a \in A, \xi \in F\}$ spans a dense submodule of F;*
   (c) *$v_g W = W u_g$ for all $g \in G$,*

   then there is a unitary operator U in $L_B(E_\rho, F)$ such that

1. (a) i. *$\Phi(a)U = U\Phi_\rho(a)$ for all $a \in A$;*
   ii. *$v_g U = U v_g^\rho$ for all $g \in G$;*
   iii. *$W = UV_\rho$.*

PROOF. We partition the proof into two steps.

Step 1. Suppose that B is a C^*-algebra.

1. Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit of A such that the net $\{\rho(e_\lambda)\}_{\lambda\in\Lambda}$ is strictly convergent to the identity operator on E, and let $(\Phi_\rho; V_\rho; E_\rho)$ be the KSGNS construction associated with ρ. Since ρ is continuous there is

$p \in S(A)$ and a completely positive linear map ρ_p from A_p to $L_B(E)$ such that $\rho = \rho_p \circ \pi_p$ (see, for example, the proof of Proposition 3.5 in [**2**]). By the proof of Theorem 4.6 in [**2**] we can suppose that E_ρ is the completion of the pre-Hilbert space $(A_p \otimes_{\text{alg}} E) / \mathcal{N}_{\rho_p}$, $V_\rho \xi = \lim\limits_{\lambda} \left(\pi_p(e_\lambda) \otimes \xi + \mathcal{N}_{\rho_p} \right)$ and $\Phi_\rho(a) \left(\pi_p(b) \otimes \xi + \mathcal{N}_{\rho_p} \right) = \pi_p(ab) \otimes \xi + \mathcal{N}_{\rho_p}$ for all $a, b \in A$ and for all $\xi \in E$.

Let $g \in G$. From

$$\left\langle \sum_{i=1}^{n} \pi_p(a_i) \otimes \xi_i + \mathcal{N}_{\rho_p}, \sum_{j=1}^{m} b_j \otimes \eta_j + \mathcal{N}_{\rho_p} \right\rangle_{\rho_p} = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \xi_i, \rho(a_i^* b_j) \eta_j \rangle$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \langle u_g(\xi_i), u_g \rho(a_i^* b_j) u_{g^{-1}} u_g(\eta_j) \rangle$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \langle u_g(\xi_i), \rho(\alpha_g(a_i^* b_j)) u_g(\eta_j) \rangle$$
$$= \left\langle \sum_{i=1}^{n} \pi_p(\alpha_g(a_i)) \otimes u_g(\xi_i) + \mathcal{N}_{\rho_p}, \sum_{j=1}^{m} \pi_p(\alpha_g(b_j)) \otimes u_g(\eta_j) + \mathcal{N}_{\rho_p} \right\rangle_{\rho_p}$$

for all $\xi_1, ..., \xi_n, \eta_1, ..., \eta_m \in E$, for all $a_1, ..., a_n, b_1, ..., b_m \in A$ and for all $g \in G$, we deduce that, there is a unitary operator v_g^ρ in $L_B(E_\rho)$ such that

$$v_g^\rho(\pi_p(a) \otimes \xi + \mathcal{N}_{\rho_p}) = \pi_p(\alpha_g(a)) \otimes u_g \xi + \mathcal{N}_{\rho_p}$$

for all $a \in A$ and for all $\xi \in E$. It is not difficult to check that the map $g \mapsto v_g^\rho$ from G to $L_B(E_\rho)$ is a unitary representation of G on E_ρ.

To show that $(\Phi_\rho, v^\rho, E_\rho)$ is a covariant representation of (G, A, α) it remains to prove that $\Phi_\rho(\alpha_g(a)) = v_g^\rho \Phi_\rho(a) v_{g^{-1}}^\rho$ for all $g \in G$ and $a \in A$. Let $g \in G$ and $a \in A$. We have

$$\begin{aligned}
(v_g^\rho \Phi_\rho(a) v_{g^{-1}}^\rho)(\pi_p(b) \otimes \xi + \mathcal{N}_{\rho_p}) &= (v_g^\rho \Phi_\rho(a))(\pi_p(\alpha_{g^{-1}}(b)) \otimes u_{g^{-1}} \xi + \mathcal{N}_{\rho_p}) \\
&= v_g^\rho(\pi_p(a \alpha_{g^{-1}}(b)) \otimes u_{g^{-1}} \xi + \mathcal{N}_{\rho_p}) \\
&= \pi_p(\alpha_g(a) b) \otimes \xi + \mathcal{N}_{\rho_p} \\
&= (\Phi_\rho(\alpha_g(a)))(\pi_p(b) \otimes \xi + \mathcal{N}_{\rho_p})
\end{aligned}$$

for all $b \in A$ and for all $\xi \in E$. Hence $\Phi_\rho(\alpha_g(a)) = v_g^\rho \Phi_\rho(a) v_{g^{-1}}^\rho$.

By Theorem 4.6 (1), [**2**] the conditions (a) and (b) are verified. To show that the condition (c) is verified, let $\xi \in E$ and $g \in G$. Then we have

$$\left\| v_g^\rho V_\rho \xi - V_\rho u_g \xi \right\|^2 = \lim_{\lambda \in \Lambda} \left\| v_g^\rho \left(\pi_p(e_\lambda) \otimes \xi + \mathcal{N}_{\rho_p} \right) - V_\rho u_g \xi \right\|^2$$
$$= \lim_{\lambda \in \Lambda} \left\| \langle \xi, \rho(e_\lambda^2) \xi \rangle + \langle \xi, \xi \rangle - \langle \rho(\alpha_g(e_\lambda)) u_g \xi, u_g \xi \rangle - \langle u_g \xi, \rho(\alpha_g(e_\lambda)) u_g \xi \rangle \right\|$$
$$\leq \lim_{\lambda \in \Lambda} \| \langle \xi, \rho(e_\lambda) \xi \rangle + \langle \xi, \xi \rangle - \langle \rho(e_\lambda) \xi, \xi \rangle - \langle \xi, \rho(e_\lambda) \xi \rangle \|$$
$$= \lim_{\lambda \in \Lambda} \| \langle \xi - \rho(e_\lambda) \xi, \xi \rangle \| = 0.$$

Therefore the condition (c) is also verified.

2. By Theorem 4.6 (2), [**2**] there is a unitary operator U in $L_B(E_\rho, F)$ defined by $U(\Phi_\rho(a)V_\rho\xi) = \Phi(a)W\xi$ such that $\Phi(a)U = U\Phi_\rho(a)$ for all $a \in A$, and $W = UV_\rho$.

Let $g \in G$. From

$$\begin{aligned}(v_gU)(\Phi_\rho(a)V_\rho\xi) &= v_g(\Phi(a)W\xi) = \Phi(\alpha_g(a))v_gW\xi \\ &= \Phi(\alpha_g(a))Wu_g\xi = U(\Phi_\rho(\alpha_g(a))V_\rho u_g\xi) \\ &= U(\Phi_\rho(\alpha_g(a))v_g^\rho V_\rho\xi) = (Uv_g^\rho)(\Phi_\rho(a)V_\rho\xi).\end{aligned}$$

for all $a \in A$ and for all $\xi \in E$, we conclude that $v_gU = Uv_g^\rho$ and thus the assertion 2. is proved.

Step 2. The general case.

Let $q \in S(B)$. Then $\rho_q = (\pi_q)_*\circ\rho$ is a $u^{(q)}$ -covariant, non-degenerate, continuous completely positive linear map from A to $L_{B_q}(E_q)$, $((\pi_q)_* \circ \Phi, v^{(q)}, F_q)$ is a covariant representation of (G, A, α) and $(\pi_q)_*(W)$ is an element in $L_{B_q}(E_q, F_q)$ such that the conditions (a), (b) and (c) from 2. are verified. By Step 1, there is a covariant representation $(\Phi_{\rho_q}, v^{\rho_q}, E_{\rho_q})$ of (G, A, α) and an element V_{ρ_q} in $L_{B_q}(E_q, E_{\rho_q})$ which verify the conditions (a), (b) and (c) from 1. and there is a unitary operator U_q in $L_{B_q}(E_{\rho_q},F_q)$ which verifies the conditions $i)$, $ii)$ and $iii)$ from 2.

Let $(\Phi_\rho; V_\rho; E_\rho)$ be the KSGNS construction associated with ρ. According to the proof of Theorem 4.6 in [**2**] , $(\pi_q)_* \circ \Phi_\rho = \Phi_{\rho_q}$; $(\pi_q)_*(V_\rho) = V_{\rho_q}$; $(E_\rho)_q = E_{\rho_q}$ for all $q \in S(B)$ and $(U_q)_q$ is a coherent sequence in $L_{B_q}(E_{\rho_q}, F_q)$. It is not difficult to check that for each $g \in G$, $(v_g^{\rho_q})_q$ is a coherent sequence in $L_{B_q}(E_{\rho_q})$, and the map $g \mapsto v_g^\rho$, where v_g^ρ is an element in $L_B(E_\rho)$ such that $(\pi_q)_*\left(v_g^\rho\right) = v_g^{\rho_q}$ for all $q \in S(B)$ is a unitary representation of G on E_ρ. Also it is not difficult to check that $(\Phi_\rho, v^\rho, E_\rho)$ is a covariant representation of (G, A, α) which verifies the conditions (a), (b) and (c) from 1.

Let $U \in L_B(E_\rho, F)$ such that $(\pi_q)_*(U) = U_q$ for all $q \in S(B)$. Clearly U is a unitary operator in $L_B(E_\rho, F)$ and it verifies the conditions $i)$, $ii)$ and $iii)$ from 2. q.e.d.

Remark 3.7 *The covariant representation $(\Phi_\rho, v^\rho, E_\rho)$ of (G, A, α) induced by ρ is unique up to unitary equivalence.*

From Proposition 3.4 and Theorem 3.6 we obtain the following corollary.

Corollary 3.8 *Let (G, A, α) be a locally C^*-dynamical system such that α is an inverse limit action, let u be a unitary representation of G on a Hilbert module*

E over a locally C^*-algebra B, and let ρ be a u -covariant, non-degenerate, continuous completely positive linear map from A to $L_B(E)$. Then ρ induces a non-degenerate representation of the crossed product $A \times_\alpha G$ on a Hilbert B -module.

The following proposition is a generalization of Proposition 2 in [**4**] in the context of locally C^*-algebras.

Proposition 3.9 *Let (G, A, α) be a locally C^*-dynamical system such that α is an inverse limit action, let B be a locally C^*-algebra, let E be a Hilbert B -module and let u be a unitary representation of G on E. If ρ is a u -covariant, non-degenerate, continuous completely positive linear map from A to $L_B(E)$, then there is a unique completely positive linear map φ from $A \times_\alpha G$ to $L_B(E)$ such that*

$$\varphi(f) = \int_G \rho(f(g))u_g dg$$

for all $f \in C_c(G, A)$. Moreover, φ is non-degenerate.

PROOF. By Theorem 3.6 there is a covariant representation $(\Phi_\rho, v^\rho, E_\rho)$ of (G, A, α) and an element V_ρ in $L_B(E, E_\rho)$ such that $\rho(a) = V_\rho^* \Phi_\rho(a) V_\rho$ and $v_g^\rho V_\rho = V_\rho u_g$ for all $a \in A$, and for all $g \in G$.

Let $\Phi_\rho \times v^\rho$ be the representation of $A \times_\alpha G$ associated with $(\Phi_\rho, v^\rho, E_\rho)$. We define $\varphi : A \times_\alpha G$ to $L_B(E)$ by

$$\varphi(x) = V_\rho^* (\Phi_\rho \times v^\rho)(x) V_\rho.$$

Clearly φ is a continuous completely positive linear map from $A \times_\alpha G$ to $L_B(E)$. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit for $A \times_\alpha G$ and let $\xi \in E$. Since $\Phi_\rho \times v^\rho$ is non-degenerate, by Proposition 4.2 in [**2**]

$$\lim_\lambda V_\rho^* (\Phi_\rho \times v^\rho)(e_\lambda) V_\rho \xi = V_\rho^* V_\rho \xi = \xi.$$

This implies that the net $\{\rho(e_\lambda)\}_{\lambda \in \Lambda}$ converges strictly to the identity map on E, and so φ is non-degenerate.

For $f \in C_c(G, A)$ we have

$$\begin{aligned} \varphi(f) &= V_\rho^* (\Phi_\rho \times v^\rho)(f) V_\rho = \int_G V_\rho^* \Phi_\rho(f(g)) v_g^\rho V_\rho dg \\ &= \int_G V_\rho^* \Phi_\rho(f(g)) V_\rho u_g dg = \int_G \rho(f(g)) u_g dg \end{aligned}$$

and since $C_c(G, A)$ is dense in $A \times_\alpha G$, φ is unique with this property. q.e.d.

Corollary 3.10 *Let (G, A, α) be a locally C^*-dynamical system, let B be a locally C^*-algebra, let E be a Hilbert B-module, let u be a unitary representation of G on E, and let ρ be a u-covariant, non-degenerate, continuous completely positive linear map from A to $L_B(E)$. If G is a compact group, then there is a unique completely positive linear map φ from $A \times_\alpha G$ to $L_B(E)$ such that*

$$\varphi(f) = \int_G \rho(f(g))u_g dg$$

for all $f \in C_c(G, A)$. Moreover, φ is non-degenerate.

PROOF. The corollary follows from Proposition 3.9, since G is compact and then α is an inverse limit action of G on A [**12**, Lemma 5.2].q.e.d.

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Department of Mathematics, Faculty of Chemistry, University of Bucharest, Bd. Regina Elisabeta nr.4-12, Bucharest, Romania
mjoita@fmi.unibuc.ro